An Electromechanical Representation of a Piezoelectric Crystal Used as a Transducer *

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THE equivalent electrical representation of a piezoelectric crystal when used as an element in an electrical circuit has been discussed by several investigators,[1] who have arrived at the circuit shown on Fig. 1. Apparently, however, no circuit has been evolved for repre-

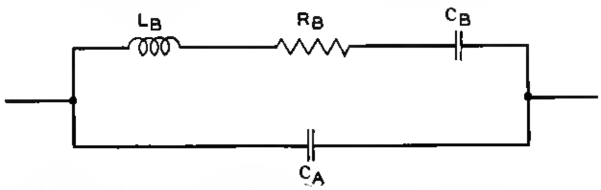

Fig. 1—Electrical representation of a piezoelectric crystal.

senting a crystal when it is used as a transducer to couple electrical circuits to mechanical systems. Since such crystals[2] are used in loud speakers, microphones, supersonic radiators, and other apparatus, it is a matter of importance to obtain such a representation. This paper discusses such an equivalent circuit and relates the elements to the mechanical, electrical, and piezoelectric constants of the material. When used as a purely electrical circuit, this representation reduces to that of Fig. 1.

When piezoelectric crystals are used to drive external mechanical systems, the modes of motion most often used are longitudinal vibrations perpendicular or parallel to the applied electric field. Accordingly, the elements of the equivalent network are derived for these cases only. The network can, however, represent any type of motion driving a load just as the network of Fig. 1 can represent the crystal for any type of motion.

Let us consider first the case of a crystal vibrating perpendicularly to

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[1] W. G. Cady, *Phys. Rev.*, **XXIX**, 1 (1922); *Proc. I. R. E.*, **X**, 83 (1922). K. S. VanDyke, Ab. 52, *Phys. Rev.*, June, 1925; *Proc. I. R. E.*, June, 1928. D. W. Dye, *Proc. Phys. Soc.* (*London*), **XXXVIII**, (5), pp. 399–453. P. Vigoreaux, *Phil. Mag.*, December, 1928, pp. 1140–53.

[2] A. M. Nicolson, *Proc. A. I. E. E.*, 38, 1315–1333, 1919. E. B. Sawyer, *Proc. I. R. E.*, **19**, No. 11, p. 2020, November, 1931. S. Ballantine, *Proc. I. R. E.*, **21**, No. 10, p. 1399, October, 1933.

the direction of the applied field. Two subdivisions of this case are usually of interest, the first when the crystal is supported at its center and drives two symmetrical loads, and the second when the crystal is supported on one end and drives a load on the other end. The symmetrical case is considered first.

By employing the well known analogies between electrical and mechanical systems, it is possible to obtain a simple network, expressed in terms of electrical symbols, which represents the properties of a piezoelectric crystal. In this representation, force is the analogue of voltage, mechanical displacement of electrical charge, and velocity of electrical current. When the electrodes are attached to the crystal faces, the equivalent network of the crystal is shown by Fig. 2, The voltage E

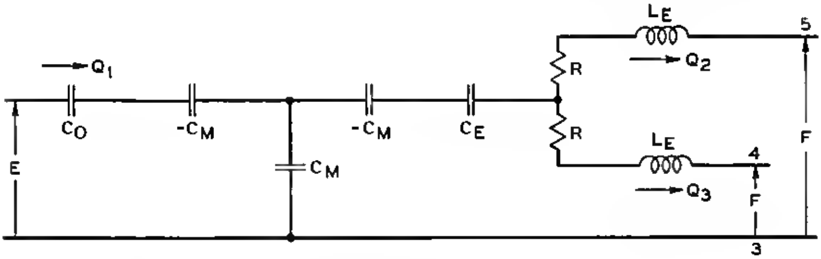

Fig. 2—Electromechanical representation of a symmetrical piezoelectric crystal.

is the voltage applied across the plates of the crystal, the force F is the force applied to each end of the symmetrical crystal, Q_1 is the electrical charge flowing in the wires connected to the crystal and Q_2 and Q_3 are the mechanical displacements of the ends of the crystal which are equal on account of the symmetry of the crystal.

The constants of the crystal can be evaluated by considering limiting cases. The capacitance C_0 is the electrostatic capacitance of the crystal clamped. The compliance C_E is the mechanical compliance of the crystal. In c.g.s. electrostatic and mechanical units, these have the values

$$C_0 = \frac{K l_m l_0}{4\pi l_e}; \qquad C_E = \frac{s l_m}{l_e l_0}, \tag{1}$$

where K is the dielectric constant of the crystal clamped, l_e the dimension of the crystal in centimeters perpendicular to the surfaces of the electrodes, l_m the length of the crystal in the direction of vibration, l_0 the length of the third axis, and s the modulus of compliance of the crystal (the inverse of Young's modulus) along the axis of vibration. The inductance L_E represents the mass reaction of half the crystal. At low

frequencies this will be equal to half the mass of the crystal, but at higher frequencies will be less due to the fact that the crystal does not move as a whole.[3] In order to resonate with the compliance C_E at the mechanical resonance frequency of the crystal, L_E must equal

$$L_E = \frac{2 l_e l_m l_0 \rho}{\pi^2}, \tag{2}$$

where ρ is the density of the crystal. The resistances R shown include the dissipation due to internal friction, supersonic radiation from the ends of the crystal, friction at the point of support and all other sources of dissipation.

If F_0 is the force required to keep the crystal from expanding when an electric charge Q_1 is applied to the crystal then C_M the mutual capacitance-compliance of the crystal is equal to

$$C_M = Q_1 / F_0. \tag{3}$$

Similarly if E_0 is the open circuit voltage for a given expansion $(Q_2 + Q_3)$ of the crystal then

$$C_M = \frac{Q_2 + Q_3}{E_0}. \tag{4}$$

In order to evaluate C_M in terms of the piezo-electric coefficient d, it is necessary to find the displacement for a free crystal when an electrical potential is applied to the crystal. Short circuiting the network of Fig. 2 on the mechanical ends and setting $F = 0$, we find

$$Q_2 + Q_3 = \frac{E\left[\dfrac{C_0 C_E}{C_M}\right]}{1 - \dfrac{C_0 C_E}{C_M^2}} = \frac{Ek\sqrt{C_0 C_E}}{1 - k^2}, \tag{5}$$

where k is the coefficient of coupling between the electrical and mechanical system is defined by the equation

$$k = \sqrt{C_0 C_E} / C_M. \tag{6}$$

Use is now made of the piezo-electric equation

$$e = dV, \tag{7}$$

[3] Strictly speaking the value of C_E is also a function of frequency, but at the first resonance it can be shown that it differs from its static value by the factor $8/[\pi^2 - k^2(\pi^2 - 8)]$ and $L_E = l_e l_m l_0 \rho / 4$. For a highly coupled crystal, this factor does not differ much from unity, and hence in the interest of simplicity, the variations of C_E have been neglected.

where e is the strain (elongation per unit length) produced in the crystal by an applied potential gradient V. Comparing (5) with (7) and noting that $Q_2 + Q_3 = el_m$, and $V = E/l_c$, we have on the insertion of the values for C_0 and C_E from (1)

$$d = \frac{k\sqrt{\frac{Ks}{4\pi}}}{1 - k^2}. \tag{8}$$

Solving for k we find

$$k = \frac{1}{2d}\sqrt{\frac{Ks}{4\pi}}\left[-1 + \sqrt{\frac{1 + 16\pi d^2}{Ks}}\right] \doteq d\sqrt{\frac{4\pi}{Ks}}, \tag{9}$$

when $16\pi d^2/Ks$ is a small quantity as it is for quartz.

When the crystal is used as an element in an electrical network, and allowed to vibrate freely, the force F of Fig. 2 can be set equal to zero and the network short-circuited. Solving for the impedance on the electrical side we find

$$Z_C = \frac{-j(1 - k^2)}{2\pi f C_0}\left[\frac{1 - f^2/f_1^2 + j/q(1 - k^2)}{1 - f^2/f_2^2 + j/q}\right], \tag{10}$$

where $f_2^2 = f_A^2$; $f_1^2 = f_A^2(1 - k^2)$ (f_A being the natural mechanical resonance frequency of the crystal), and q is the ratio of the reactance of the condenser C_E to the resistance $R/2$ or

$$q = 2/2R\pi f_A C_E. \tag{11}$$

It is easily shown that the impedance Z_c is also the impedance of the network of Fig. 1 if [4]

$$\begin{aligned} C_A &= C_0; \\ C_B &= C_0k^2/(1 - k^2); \\ L_B &= 1/4\pi^2 f_A^2 k^2 C_0; \\ R_B &= 1/2\pi f_A C_0 k^2 q. \end{aligned} \tag{12}$$

Hence the representation in Fig. 2 reduces to the well known Fig. 1, when the crystal is free to vibrate.

A network representing the second case, when one end is supported with the other end used to drive a load, is shown on Fig. 3. The method of deriving the constants is the same and all of the constants

[4] If account is taken of the variation of C_E and L_E with frequency, the elements are $C_A = C_o$; $C_B = \frac{8k^2C_o}{\pi^2(1 - k^2)}$; $L_B = \frac{1}{32k^2f_A^2C_o}$; $R_B = 1/2\pi f_A C_o k^2 q$ where $q = \frac{1}{\pi R C_E f_A}$.

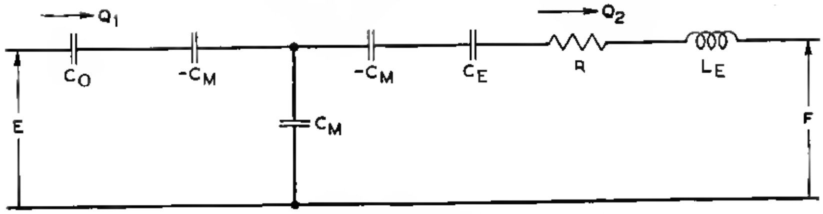

Fig. 3—Electromechanical representation of a piezoelectric crystal clamped on one end.

are the same except L_E, which is twice as large since twice the mass is moved from the clamping position.

When the direction of motion is parallel to the direction of the applied field, the same networks hold but the elements have different lengths entering into their determinations. The direction of the applied field and of the motion is designated by l_e. The other two axes are still designated by l_m and l_0. The resulting constants are

$$C_0 = \frac{Kl_m l_0}{4\pi l_e}; \qquad C_E = \frac{sl_e}{l_m l_0}; \qquad C_M \doteq \frac{Ks}{4\pi d};$$
$$L_{E_1} = \frac{2\rho l_e l_0 l_m}{\pi^2}; \qquad L_{E_2} = \frac{4\rho l_e l_0 l_m}{\pi^2}, \tag{13}$$

where L_{E_1} is the mechanical inductance for the symmetrical case (Fig. 2) and L_{E_2} for the dissymmetrical case (Fig. 3).

A simple example of the use of Fig. 2 in determining the effect of a mechanical load on the impedance of a crystal is the problem of finding out how much the energy radiated by a crystal to the surrounding air affects the decrement of a quartz crystal vibrating longitudinally. When a crystal vibrates, energy is radiated to the surrounding medium by the motion of the ends of the crystal. If the dimensions of the ends of the crystal are comparable to a wave-length or greater—which they will ordinarily be for a crystal vibrating at a high frequency—it is well known [5] that the radiating surface experiences a resistance to motion equal to

$$R_R = \rho_A b \text{ (mechanical ohms)} \tag{14}$$

per square centimeter, where ρ_A is the density of the medium and b the velocity of propagation. For air R_R is about 41 ohms per square centimeter. Hence the equivalent circuit for a crystal vibrating in air is Fig. 2 terminated at the terminals 3–4 and 3–5 by the mechanical

[5] See "Theory of Vibrating Systems and Sound," I. B. Crandall, Chap. 4, D. Van Nostrand Co.

resistances

$$R_R = 41 l_e l_0. \tag{15}$$

If all other sources of dissipation were eliminated, the radiation resistance would produce a limiting value for the decrement of a crystal which may be calculated as follows. From equation (11), the value of q for the mechanical system is

$$q = \frac{1}{\pi f_A R_R C_E} = \frac{2\sqrt{\rho/s}}{41\pi} \tag{16}$$

on inserting the values of R_R, C_E, and $f_A = 1/2l_m\sqrt{\rho s}$. Since $\rho = 2.65$ and $s = 1.27 \times 10^{-12}$ for a perpendicularly cut quartz crystal

$$q = 2.24 \times 10^4. \tag{17}$$

The decrement of a crystal in terms of the circuit of Fig. 1 is

$$\delta = \frac{R_B}{2f_A L_B}. \tag{18}$$

Inserting the values of footnote (4) we find

$$\delta = \frac{8}{\pi q} = 1.14 \times 10^{-4}. \tag{19}$$

Van Dyke [6] has measured the limiting value of the decrement of a perpendicularly cut quartz crystal vibrating in air and finds it to be 1.26×10^{-4}. Since the residual losses were about 5 per cent of the radiation losses, this agrees well with the value found in equation (19).

The equivalent circuits of Figs. 2 and 3 may also be used as the basis of design for mechanical systems, such as loud speakers, microphones and supersonic radiators.

[6] *Proc. I. R. E.*, April, 1935.